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Relation between covariant and parafermionic construction of affine Kac–Moody algebras

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Abstract. From the extension of the covariant vertex operator construction for the affine Kac–Moody algebras to Lorentzian algebras, we show that the parafermionics realization of arbitrary level affine algebras can be interpreted in terms of bosonic realization. This connection is explicitly illustrated for level $k = 2$ $\widehat{SU}(2)$ algebra.

1. Introduction

The vertex operators representation of Kac–Moody (KM) Lie algebras (for a review on the subject and references see [1]) is a mighty theoretical tool in conformal field theories with applications in statistical systems and string models. As is well known the explicit form of the vertex is strongly dependent on the level k of the representation.

In [2] Goddard and Olive have suggested a vertex-covariant construction by means of which it is possible to obtain simply-laced finite-dimensional Lie algebras, affine Kac–Moody algebras or Lorentzian algebras by considering respectively the Euclidean, singular or Lorentzian lattice [3]. Their construction in the case of affine algebras is very interesting because it is independent of the level k, in the sense that it makes use of the same fields for any k value.

Another possibility is to use the Frenkel–Kac–Segal construction [4] which is interpreted as a transverse or light-gauge construction and gives only the $k = 1$ representation. A generalization of this construction for $k > 1$ is obtained by the use of parafermionic fields [5–7], but this realization is dependent on k.

The aim of this work is to clarify a few aspects of the connection between the covariant and parafermionic construction. In section 2 we recall the covariant construction of affine KM algebras and extend it to Lorentzian algebras by adding an extra light-like direction. In section 3 we discuss the relation between the construction for $k \neq 1$ of the affine algebras which we have recalled in section 2 and the parafermionic construction. We show explicitly the equivalence between parafermionic fields and our fields in the case of $SU(2)$ for $k = 2$.

2. Covariant construction of KM algebras

The Goddard–Olive construction of affine KM algebras can be interpreted as a vertex construction of a Lie algebra in a singular lattice obtained by adding a light-like direction

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to the Euclidean lattice defining the horizontal finite Lie algebra. The outer derivation is consistent with the extension of the singular lattice to a Lorentzian one and with the interpretation of the affine algebra as a sub-algebra of a Lorentzian one.

Let us recall shortly the essential steps of the covariant construction.

We introduce an infinite set of annihilation and creation operators a_n^μ $n \in Z$, satisfying the commutation relations

$$\left[a_n^\mu, a_m^\nu\right] = n g^{\mu\nu} \delta_{n+m,0} \tag{1}$$

with $g^{\mu\nu}$ a Minkowskian diagonal metric and

$$a_n^{\mu\dagger} = a_{-n}^\mu . \tag{2}$$

The momentum operator is $a_0^\mu = p^\mu$ and we introduce the q^μ operator such that

$$\left[q^\mu, p^\nu\right] = \mathrm{i} g^{\mu\nu} . \tag{3}$$

Then we introduce the fields

$$Q^\mu(z) = q^\mu - \mathrm{i} p^\mu \ln z + \mathrm{i} \sum_{n \neq 0} \frac{a_n^\mu}{n} z^{-n} \tag{4}$$

and

$$Q^{\mu(1)}(z) = \mathrm{i} \frac{\mathrm{d}}{\mathrm{d}z} Q^\mu(z) = \sum_n a_n^\mu z^{-n-1} . \tag{5}$$

If we consider the roots r belonging to a Lorentzian lattice, we can decompose them as

$$r = \alpha + nK^+ + mK^- \tag{6}$$

with $K^{\pm 2} = 0$, $K^+ \cdot K^- = 1$ and α belonging to Λ, the horizontal Euclidean lattice of a simply-laced algebra.

Then the vertex operatc. associated to a root r is

$$U^r(z) =: \mathrm{e}^{\mathrm{i} r \cdot Q(z)} : \tag{7}$$

where the dots indicate the normal ordering, with the usual property

$$U^{r\dagger}(z) = U^{-r}\left(\frac{1}{z^*}\right) . \tag{8}$$

The affine sub-algebra is spanned, for the real roots, by

$$A^{\alpha+nK^+} = \frac{c_\alpha}{2\pi \mathrm{i}} \oint \mathrm{d}z \, U^{\alpha+nK^+}(z) \tag{9}$$

c_α being a co-cycle, and for the imaginary roots by

$$H^i_{nK^+} = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z : Q^{i(1)}(z) U^{nK^+}(z) : \tag{10}$$

were the i, j indices are restricted to the Euclidean lattice.

A quite general construction of co-cycles has been done by Goddard–Olive [2] and we can use their co-cycles in our construction, so we have not to worry about this aspect.

The commutation relations are

$$\left[H^i_{nK^+}, H^j_{mK^+}\right] = n\delta^{ij}\delta_{n+m,0}K^+ \cdot p \tag{11}$$

for the Heisenberg sub-algebra and

$$\left[A^{\alpha+nK^+}, A^{\beta+mK^+}\right] = 0 \qquad \alpha\cdot\beta \geqslant 0 \tag{12}$$

$$\left[A^{\alpha+nK^+}, A^{\beta+mK^+}\right] = \epsilon(\alpha,\beta)A^{\alpha+\beta+(n+m)K^+} \qquad \alpha\cdot\beta = -1 \tag{13}$$

$$\left[A^{\alpha+nK^+}, A^{\beta+mK^+}\right] = \alpha\cdot H_{(n+m)K^+} + n\delta_{n+m,0}K^+\cdot p \qquad \alpha = -\beta \tag{14}$$

$$\left[H^i_{nK^+}, A^{\alpha+mK^+}\right] = \alpha^i A^{\alpha+(n+m)K^+}. \tag{15}$$

We can also define a derivation (not belonging to the affine KM algebra) by

$$D = -K^- \cdot p \tag{16}$$

with the commutators

$$\left[D, A^{\alpha+nK^+}\right] = -nA^{\alpha+nK^+} \tag{17}$$

$$\left[D, H^i_{nK^+}\right] = -nH^i_{nK^+}. \tag{18}$$

One notes that $K^+\cdot a_n$ commutes with any element of the algebra; therefore we can take them to be constant and in particular

$$K^+\cdot p \to k \qquad K^+\cdot a_n \to 0 \qquad \text{if} \quad n \neq 0. \tag{19}$$

The level independence is now evident from the above construction and from (14) and (19).

Let us emphasize that this property is a natural consequence of the extension of the Euclidean lattice to a Lorentzian one.

3. Relationship with parafermions

The choice of (19) corresponds to making a transformation from the covariant gauge to the transverse gauge and we will examine this correspondence.

In this transformation the vertex operator $U^{nK^+}(z)$ is reduced to

$$U^{nK^+}(z) = \mathrm{e}^{\mathrm{i}nK^+\cdot q} z^{nK^+\cdot p} \tag{20}$$

where the phase $\mathrm{e}^{\mathrm{i}nK^+\cdot q}$ is irrelevant in this context.

The other operators become

$$U^{\alpha+nK^+}(z) = z^{nk} : \mathrm{e}^{\mathrm{i}\alpha\cdot Q(z)} : \tag{21}$$

$$A^\alpha_n = \frac{c_\alpha}{2\pi\mathrm{i}}\oint \mathrm{d}z\, z^{nk}U^\alpha(z) \tag{22}$$

$$: Q^{i(1)}(z)U^{nK^+}(z) := z^{nk}Q^{i(1)}(z) \tag{23}$$

$$H^i_n = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\, z^{nk}Q^{i(1)}(z). \tag{24}$$

If $k = 1$ this is the Frenkel–Kac–Segal construction but if $k > 1$ it looks like a different new construction. Now we can discuss the connection of this construction with the parafermionic one.

For $k > 1$ the Fock space F^k of Heisenberg sub-algebra is build up by the set of H_n^i operators, which is the subset of the creation and annihilation operators of whole Fock space F, a_m^μ, with $\mu = i$ and $m = nk$ with k fixed.

Then we decompose the F space in

$$F = F^k \otimes \Omega^k$$

where Ω^k is the vector space of vacuum vectors for the Heisenberg sub-algebra.

The Hilbert space where the operators of (4) act is

$$H = F \otimes \Lambda^* .$$

(Λ^* is the the dual of Λ lattice.)

Let us now define the fields on $F^k \otimes \Lambda^*$ space:

$$H^i(z) = \sum_n a_{nk}^i z^{-nk-1} \tag{25}$$

(for $k = 1$, from (5) $H^i(z) = Q^{i(1)}(z)$) with

$$H^i(z)H^j(\xi) =: H^i(z)H^j(\xi) : +k\delta_{ij}\frac{z^{k-1}\xi^{k-1}}{(z^k - \xi^k)^2} \tag{26}$$

and

$$X^i(z) = q^i - \mathrm{i}p^i \ln z + \mathrm{i}\sum_{n\neq 0} \frac{a_{nk}^i}{kn} z^{-kn} . \tag{27}$$

We can also define a vertex operator on $F^k \otimes \Lambda^*$:

$$\mathcal{U}^\alpha(z) = z^{1-1/k} : \mathrm{e}^{\mathrm{i}\alpha\cdot X(z)} : \tag{28}$$

which satisfies the relation

$$\mathcal{U}^\alpha(z)\mathcal{U}^\beta(\xi) =: \mathcal{U}^\alpha(z)\mathcal{U}^\beta(\xi) : (z^k - \xi^k)^{\alpha\cdot\beta/k} . \tag{29}$$

Morever, the fields on Ω^k are defined by

$$\psi^\alpha(z) = z^{1/k-1} : \mathrm{e}^{\mathrm{i}\alpha\cdot(Q(z)-X(z))} : \tag{30}$$

and satisfy

$$\psi^\alpha(z)\psi^\beta(\xi) =: \psi^\alpha(z)\psi^\beta(\xi) : \frac{(z-\xi)^{\alpha\cdot\beta}}{(z^k - \xi^k)^{\alpha\cdot\beta/k}} . \tag{31}$$

We shall show that these fields can be interpreted as the parafermionic fields for the k level.

In fact, by means of the conformal transformation $z \to z^k$ and using the isomorphism

$$a_{nk}^i \longrightarrow \sqrt{k} a_n^i \qquad q^i \longrightarrow \frac{1}{\sqrt{k}} q^i \tag{32}$$

we can write (27) as

$$\phi^i(z) = q^i - \mathrm{i}p^i \ln z + \mathrm{i}\sum_{n\neq 0}\frac{a_n^i}{n}z^{-n} \tag{33}$$

and the vertex $\mathcal{U}^\alpha(z)$ of (28) becomes

$$\mathcal{U}^\alpha(z) =: \mathrm{e}^{\mathrm{i}\alpha\phi(z)/\sqrt{k}} : . \tag{34}$$

The above equations define the quantities appearing in the Gepner construction in terms of these used in the Lorentzian lattice approach.

In fact the currents $h^i(z)$ and $\chi_\alpha(z)$ given by equation (2) of [8] can be written as

$$h^i(z) = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}z}\phi^i(z) = \sqrt{k}\sum_n a_n^i z^{-n-1} \tag{35}$$

$$\chi_\alpha(z) = c_\alpha \psi^\alpha(z)\mathcal{U}^\alpha(z) \tag{36}$$

and they satisfy the commutation relations of equations (4) of [8].

By the definition (30) and by (31), using (32)–(36) we obtain the parafermionics OPE relations

$$\psi^\alpha(z)\psi^\beta(\xi) =: \psi^\alpha(z)\psi^\beta(\xi) : \prod_{p=1}^{k-1}(z^{1/k} - \epsilon^p\xi^{1/k})^{-\alpha\cdot\beta}(z-\xi)^{\alpha\cdot\beta(1-\frac{1}{k})} \tag{37}$$

where $\epsilon = \mathrm{e}^{2\pi\mathrm{i}/k}$.

For $\alpha = -\beta$ these relations are

$$\psi^\alpha(z)\psi^{-\alpha}(\xi) = k^2(z-\xi)^{-2+2/k} \tag{38}$$

and for $\alpha\cdot\beta = -1$

$$\psi^\alpha(z)\psi^\beta(\xi) = k(z-\xi)^{-1-\alpha\cdot\beta/k}\psi^{\alpha+\beta}(\xi) . \tag{39}$$

Therefore by comparing (38) and (39) with equation (6) of [8] we can states that the fields $\psi_\alpha(z)$, given by (30) and (31), after a conformal transformation and the use of the isomorphism of (32), are proportional to the parafermions introduced in [8].

We can decompose $\psi^\alpha(z)$ in a sum of k parts with definite boundary conditions

$$\psi^\alpha(z) = \sqrt{k}\sum_{\lambda=0}^{k-1}\psi_\lambda^\alpha(z) \tag{40}$$

where

$$\psi_\lambda^\alpha(\mathrm{e}^{2\pi\mathrm{i}}z) = \epsilon^\lambda\psi_\lambda^\alpha(z) . \tag{41}$$

In this case the new modes become

$$A_n^\alpha = \frac{c_\alpha\sqrt{k}}{2\pi\mathrm{i}}\oint \mathrm{d}z\, z^n\mathcal{U}^\alpha(z)\psi_\lambda^\alpha(z) \tag{42}$$

where the boundary conditions for parafermionic fields are selected by the relation

$$\alpha\cdot p + \lambda = 0 \qquad \mathrm{mod}\, k \tag{43}$$

which is imposed in order to have a single-valued integrand; therefore we must take a particular quotient in the Ω^k space.

In this way we obtain the discrete symmetry which is associated to the charge in $\Lambda^*/k\Lambda$.

Let us emphasize once more that the parafermions appear naturally in this procedure and they are built on the bosonic space by means of bosonic fields through (30), so we consider the present procedure to obtain parafermions as a generalized bosonization procedure.

Finally we illustrate explicitly this connection in the simplest case of $SU(2)$ affine algebra with $k = 2$. In this case

$$Q(z) - X(z) = \mathrm{i} \sum_{n \in 2Z+1} \frac{a_n}{n} z^{-n} \tag{44}$$

is an odd field and

$$\psi^{\pm}(z) = z^{-1/2} : \exp\left(\mp \sum_{n \in 2Z+1} \frac{a_n}{n} z^{-n} \right) : \tag{45}$$

satisfy

$$\psi^{\pm}(z)\psi^{\mp}(\xi) =: \psi^{\pm}(z)\psi^{\mp}(\xi) : \left(\frac{z+\xi}{z-\xi} \right). \tag{46}$$

The vertex operator

$$A_n^{\pm} = \frac{c_\alpha}{2\pi i} \oint \mathrm{d}z\, z^{2n} \mathcal{U}^{\pm}(z) \psi^{\pm}(z) \tag{47}$$

is an odd or even field if $\alpha \cdot p \in 2Z + 1$ or $\alpha \cdot p \in 2Z$ respectively.

$\psi^{\pm}(z)$ contains odd and even modes and we can write them as the sum of two fermionic fields of defined parity (type Ramond (R) or Neveu–Schwarz (NS)) [9]:

$$\psi^{\pm}(z) = \sqrt{2} \left(\psi^{\pm}(z^2)_{\mathrm{R}} + \psi^{\pm}(z^2)_{\mathrm{NS}} \right) . \tag{48}$$

In (47) we select the even modes, therefore $A_n^{\pm}$ takes contribution from Ramond fermions type if p belongs to a spin representation or by Neveu–Schwarz fermions if p belongs to a vectorial or scalar representation:

$$A_n^{\pm} = \frac{c_\alpha \sqrt{2}}{2\pi \mathrm{i}} \oint \mathrm{d}z\, z^{2n} \mathcal{U}^{\pm}(z) \psi^{\pm}(z^2)_{\mathrm{R(NS)}} . \tag{49}$$

By means of the isomorphism defined by (33) we obtain an equivalent construction that is directly related to standard realization [9].

If we define the free massless Bose chiral field $\phi(z)$ (equation (32)) and the currents

$$h(z) = \sqrt{2}\mathrm{i} \frac{\mathrm{d}}{\mathrm{d}z} \alpha \phi(z) \tag{50}$$

$$\chi^{\alpha}(z) = c_\alpha \sqrt{2} : \mathrm{e}^{\mathrm{i}\alpha\phi(z)/\sqrt{2}} : \psi^{\alpha}_{\mathrm{R(NS)}}(z) \tag{51}$$

where $\alpha = \pm\sqrt{2}$ and ψ is a R or NS fermion, we obtain the usual theory with a stress tensor which generates a Virasoro algebra with central charge $c = \frac{3}{2}$ corresponding to level $k = 2$ $SU(2)$ affine algebra [8, 9].

Of course one of the most interesting application of the covariant vertex construction is the realization of Lorentzian algebras [10].

The unified construction of arbitrary level representations of affine KM algebras oppears quite naturally in this context. In fact, the level k can be changed by the action of the pure Lorentzian generators. This is in complete analogy with the case of affine algebras where the weights of horizontal finite-dimensional Lie sub-algebra are changed by the action of the affine generators.

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